THE UNIVERSAL EDGE ELIMINATION POLYNOMIAL AND THE DICHROMATIC POLYNOMIAL

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ABSTRACT. The dichromatic polynomial $Z(G;q,v)$ can be characterized as the most general C-invariant, i.e., a graph polynomial satisfying a linear recurrence with respect to edge deletion and edge contraction. Similarly, the universal edge elimination polynomial $\xi(G;x,y,z)$ introduced in [2] can be characterized as the most general EE-invariant, i.e., a graph polynomial satisfying a linear recurrence with respect to edge deletion, edge contraction and edge extraction. In this paper we examine substitution instances of $\xi(G;x,y,z)$ and show that among these the dichromatic polynomial $Z(G;q,v)$ plays a distinctive rôle.

1. INTRODUCTION

Many graph polynomials satisfy linear recurrence relations with respect to various graph decomposition operations, e.g. edge or vertex deletion and contraction. Examples of such polynomials include the chromatic polynomial, the matching polynomial, and the Tutte polynomial, (cf. [3] and [6]). Given set of graph operations, it is natural to ask whether there exists a graph polynomial which satisfies a recurrence relation with respect to these operations and which contains as substitution instances all other graph polynomials which satisfy recurrence relations with respect to the same operations. Such a polynomial is called *universal* with respect to the set of graph operations.

In this extended abstract we consider multi-graphs $G=(V,E)$ with vertex set V and edge set E; $G_1 \sqcup G_2$ denotes the disjoint union of two graphs G_1 and G_2; $\varnothing$ denotes a null graph (a graph with no vertices or edges), and E_1 denotes a singleton.

Let $G=(V(G),E(G))$ be a graph. We denote by $G-e$ the graph obtained from G by deleting the edge e, by G/e the graph obtained from G by contracting e, and by $G \dagger e$ the graph obtained from G by extracting of the edge e, i.e. deleting the edge e together with its endpoints. For $A \subseteq E$ we denote by $k(A)$ the number of components of the spanning subgraph (V,A).

Definition 1.1 (EE-invariants). Let $p(G)$ be a graph polynomial or a graph invariant. We say that $p(G)$ is an *EE-invariant* if there are $\alpha,\beta,\gamma,\delta \in \mathbb{R}$, for which $p(E_1)=\delta$ and $p(\varnothing)=1$, and

$$p(G) = \alpha p(G-e) + \beta p(G/e) + \gamma p(G \dagger e) \text{ for every } e \in E(G),$$
$$p(G_1 \sqcup G_2) = p(G_1) \cdot p(G_2).$$

We say that $p(G)$ is a *C-invariant* if $\gamma = 0$, i.e., edge extraction is not needed in the recurrence relation.

It is well known that there is a universal C-invariant given by

$$U_C(G;w,q,v) = w^{|E|} \cdot Z(G;q,\frac{v}{w}) \tag{1.1}$$

where

$$Z(G;x,y) = \sum_{A\subseteq E} x^{k(A)} y^{|A|}. \tag{1.2}$$

is the the *dichromatic polynomial* (aka the Potts model in statistical mechanics).

The following theorems directly follow from the results of [2]:

Theorem 1.1 (Universality of ξ)**.** *There exists a unique EE-invariant $\xi(G;x,y,z)$, which satisfies the recurrence relation in Definition 1.1 under the substitution*

$$[\alpha \mapsto 1, \beta \mapsto y, \gamma \mapsto z, \delta \mapsto x].$$

Any other EE-invariant can be obtained from $\xi(G;x,y,z)$ by further substitution of variables x,y,z and a possible prefactor $w^{|E|}$, for some $w \in \mathbb{R}$.

Theorem 1.2. *The universal EE-polynomial $\xi(G;x,y,z)$ has a subset expansion as follows:*

$$\xi(G;x,y,z) = \sum_{(A\sqcup B)\subseteq E} x^{k(A\sqcup B)-k_{cov}(B)} \cdot y^{|A|+|B|-k_{cov}(B)} \cdot z^{k_{cov}(B)}. \tag{1.3}$$

where by slight abuse of notation we use $(A \sqcup B) \subseteq E$ for summation over subsets $A, B \subseteq E$, such that the subsets of vertices $V(A)$ and $V(B)$, covered by respective subset of edges, are disjoint: $V(A) \cap V(B) = \varnothing$.

Recall that $k(A)$ denotes the number of connected components of the spanning subgraph (V,A). $k_{cov}(B)$ denotes the number of covered connected components, i.e. the connected components of the graph $(V(B), B)$.

Another universal EE-invariant was proposed by M. Trinks in [9]: The covered component polynomial $C(G;x,y,z)$ is defined as an ordinary generating function $C(G;x,y,z) = \sum_{i,j,k\geq 0} c_{i,j,k} x^i y^j z^k$, where $c_{i,j,k}$ denotes the number of edge subsets $A \subseteq E$ of size $|A| = j$, such that the graph (V,A) has exactly i components, from which k are covered by edges (i.e. are not singletons). Trinks proves that this graph polynomial is connected to $\xi(G;x,y,z)$ by $C(G;x,y,z) = \xi(G;x,y,xy(z-1))$.

2. The substitution instances of EE-polynomials

The universal edge elimination polynomial ξ generalizes many known graph polynomials from the literature. The bivariate matching polynomial denoted here $U_M(G;x,y)$ (cf. [8]) and its substitution instances, including the acyclic (matching defect) polynomial, $\mu(G;x)$, and generating matching polynomial, $g(G;x)$, satisfy recurrence relations with respect to edge deletion and extraction. Hence, they are EE-invariants and are substitution instances of ξ.

The dichromatic polynomial $Z(G;q,v)$ is closely related to the Tutte polynomial. The chromatic polynomial $\chi(G;\lambda)$ is obtained as a substitution instance of $Z(G;q,v)$. Both Z and χ satisfy recurrence relations with respect to edge deletion and contraction, and hence they are are EE-invariants.

Another bivariate extension of the chromatic polynomial $p(g;x,y)$ was introduced in [4]. it was shown in [2] that $p(g;x,y)$ satisfies a recurrence relation with respect to edge deletion, contraction and extraction, and is therefore also an EE-polynomial. The vertex cover polynomial $\Psi(G;\lambda)$ was introduced in [5]. $\Psi(G;\lambda)$ is a substitution instance of $P(G;x,y)$, cf. [4]. In Subsection 2.1 we prove that the edge-cover polynomial $E(G;\lambda)$ is also an EE-invariant.

Figure 2.1 depicts the relations in terms of substitution of the above-mentioned EE-invariants which are graph polynomials.

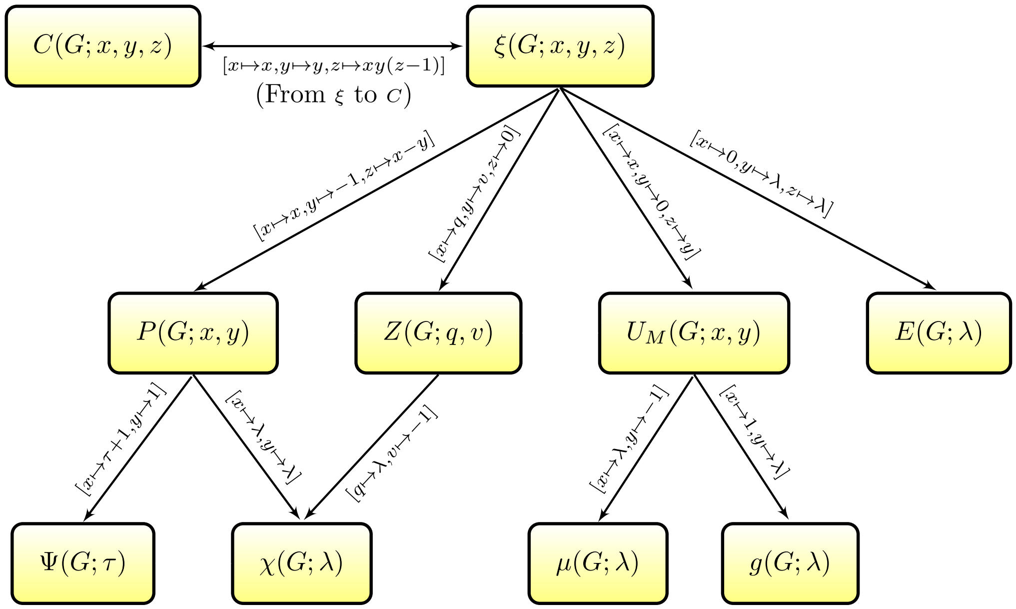

FIGURE 2.1. The various EE-polynomials and the relationship between them in terms of substitution.

The independent set polynomial is given as a substitution instance of ξ times a simple prefactor, $I(G;x) = \Psi(G;x^{-1}) \cdot x^n$, where n is the number of vertices in G.

2.1. **A new EE-invariant.** In [1] the *edge-cover polynomial* $E(G;\lambda)$ is introduced. An edge cover is a subset $A \subseteq E$ of edges such that every vertex is incident to at least one edge of A. Now the edge-cover polynomials is defined by

$$E(G;\lambda) = \sum_{\substack{A\subseteq E:\\ A \text{ is an edge cover}}} \lambda^{|A|}.$$

Proposition 2.1. *It holds that* $E(G;\lambda) = \xi(G;0,\lambda,\lambda)$.

Proof. The proof makes use of the explicit definition of $\xi(G;x,y,z)$. If we substitute $[x \mapsto 0, y \mapsto \lambda, z \mapsto \lambda]$, then all the summands that have x in a positive power disappear, so the only edge subsets A and B that contribute to the sum should satisfy $k(A \sqcup B) = k_{cov}(B)$, which holds iff $A = \emptyset$ and B is an edge cover. □

3. $\xi(G;x,y,z)$ AND THE DICHROMATIC POLYNOMIAL $Z(G;q,v)$

Among the substitution instances of $\xi(G;x,y,z)$ the dichromatic polynomial $Z(G;q,v)$ plays a special rôle, in as much as $\xi(G;x,y,z)$ can be written as a weighted sum of polynomials $Z(G[U];q,v)$, where $G[U]$ ranges over all subgraphs of G, and the weights are simple graph invariants of $G[U]$. More precisely,

Theorem 3.1. *It holds that*

$$\xi(G;x,y,z) = \sum_{B\subseteq E} y^{|B|-k_{cov}(B)} z^{k_{cov}(B)} \cdot Z(G-B;x,y) \tag{3.1}$$

where the summation is over all subsets B of the edge-set of G.

Proof. Since $V(A)$ and $V(B)$ are disjoint, the connected components in the graph $(V, A \sqcup B)$ can be partitioned into three parts: (1) isolated vertices, (2) connected components of $(V(A), A)$ and (3) connected components of $(V(B), B)$. By definition $k_{cov}(B)$ counts the connected components of part (2). Parts (1) and (3) are exactly all of the connected components in the graph $(V - V(B), A)$. Hence, the number of connected components in $(V - V(B), A)$ is $k(A \sqcup B) - k_{cov}(B)$, which we denote by $k(V - V(B), A)$. Therefore,

$$\xi(G;x,y,z) = \sum_{B\subseteq E} \sum_{A\subseteq E:\, V(A)\cap V(B)=\varnothing} x^{k(V-V(B),A)} y^{|A|+|B|-k_{cov}(B)} z^{k_{cov}(B)}.$$

Taking the factors that depend only on B out of the inner sum, we get

$$\xi(G;x,y,z) = \sum_{B\subseteq E} y^{|B|-k_{cov}(B)} z^{k_{cov}(B)} \sum_{A\subseteq E:\, V(A)\cap V(B)=\varnothing} x^{k(V-V(B),A)} y^{|A|}. \tag{3.2}$$

Now lets look at $Z(G - B; x, y)$. Comparing the inner sum of Equation (3.2) with Equation (1.2), it remains to notice that the inner sum in Equation (3.2) is over all sets of edges in G which are not incident with any vertex of $V(B)$, i.e. is indeed over all $A \subseteq E(G - B)$. □

Remark 3.2. The dichromatic polynomial is essentially, up to a simple prefactor (weight), the most general C-invariant. Similarly, by summing over all induced subgraphs, one can obtain the most general EE-invariant from the dichromatic polynomial. In other words, Equation (3.1) generalizes Equation (1.1).

Remark 3.3. Equation (3.1) also has another analog. In 1981 C. Godsil and I. Gutman showed a connection between the characteristic polynomial $\phi(G; \lambda)$ and the acyclic polynomial $\mu(G; \lambda)$. For a set of edges C in G we denote by $G - C$ the induced subgraph of G obtained by deleting the vertices incident to edges in C.

Theorem 3.4 ([7]). *It holds that*

$$\phi(G;\lambda) = \sum_{C\subseteq E} (-2)^{k_{cov}(C)}\, \mu(G-C,\lambda) \tag{3.3}$$

where the summation is over all 2-regular subgraphs of G, including $C = \varnothing$.

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